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COM205T Discrete Structures for Computing-Lecture Notes

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Problem Session - Infinite Sets

1. Determine which of the following sets are finite and which are infinite.
   (a) The set of all strings $\{a, b, c\}^*$ of length no greater than k.
   (b) The set of all m x n matrices with entries from $\{0, 1, \ldots, k\}$ where k, m, n are positive integers.
   (c) The set of all functions from $\{0,\ 1\}$ to **I**.
   (d) The set of all polynomial of degree two with integer coefficients.
2. Prove that the intersection of two infinite sets is not necessarily infinite. (the class of infinite sets is not closed under intersection)
3. Let A and B be infinite sets such that $B \subset A$. Is the set $A - B$ necessarily finite? Is it necessarily infinite? Give examples to support your answer.
4. Check whether the following sets are countable.(countable finite or countably infinite)
   (a) The set of all functions from the set $\{0, 1, \ldots, k-1\}$ to $\mathbf{N}$, where k is a fixed integer.
   (b) Let S be a countably infinite set. (i) Set of all subsets of $\mathbf{S}$ (ii) set of all finite subsets of $\mathbf{S}$.
   (c) The set of computational problems in computer science.
   (d) $A = \{(x_1, \ldots, x_k) \mid x_i \in \mathbf{N}$ and k is a fixed integer $\}$
   (e) $A = \{(x_1, \ldots, x_k) \mid x_i \in \mathbf{N}$ and k is a variable integer $\}$

Solutions:

1. (a) `The set of all strings` $\{a, b, c\}^*$ `of length no greater than` k.
Finite set (Since k is fixed). Number of strings of length zero $=1$. Number of strings of length one $=3$. Number of strings of length two $= 3^2 = 9$. Thus, number of strings of length no greater than k is $3^0 + 3^1 + \ldots + 3^k$.

(b) `The set of all` m `x` n `matrices with entries from` $\{0, 1, \ldots, k\}$ `where` k, m, n `are positive integers.`
Number of entries in a $m \times n$ matrix is mn and each entry has $k+1$ options, thus there are $(k+1)^{mn}$ possible $m \times n$ matrices if k, m and n are fixed, which is a finite set. If k is a variable, then the number of possibilities are $\sum_{k=1}^{\infty}(k+1)^{mn}$, which is a infinite set. When k, m and n are variables, it is countably infinite.

(c) `The set of all functions from {0, 1} to I.`
Countably infinite. Since, $f(0) = a$ and $f(1) = b$, where (a, b) is $I \times I$, which is countably infinite.
(d) `The set of all polynomial of degree two with integer coefficients.`
Countably infinite. Since, counting the set of all polynomials with integer co-efficients is equivalent to count $I \times I \times I$. i.e., $\{(a_1, a_2, a_3) \mid a_i$ is a co-efficient $\}$ and the number of such (a_1, a_2, a_3) is $I \times I \times I$

2. Prove that the intersection of two infinite sets is not necessarily infinite. (the class of infinite sets is not closed under intersection)
   Let $A = I^+ \cup \{0\}$ and $B = I^+ \cup \{0\}$. Clearly, A and B are infinite sets. $A \cap B = \{0\}$, which is finite.
3. Let A and B be infinite sets such that $B \subset A$. Is the set $A - B$ necessarily finite? Is it necessarily infinite? Give examples to support your answer.
   It can be either finite or infinite. For example: If $A = I$ and $B = N$ are the infinite sets then $A - B$ is infinite. If $A = N$ and $B = I^+$ are the infinite sets then $A - B$ is finite.
4. (a) The set of all functions from the set $\{0, 1, \ldots, k-1\}$ to N, where k is a fixed integer.
   $f : \{0, 1, \ldots, k-1\} \rightarrow N$ such that $f(0) = a_1, f(1) = a_2$, ..., $f(k-1) = a_k$, where $a_i \in N$. Thus the problem is equivalent to finding the number of elements in $N \times N \times \ldots \times N(k$ times), which is countably infinite.

   (b) Let S be a countably infinite set. (i) Set of all subsets of S (ii) set of all finite subsets of S.

   - (i): Since the power set of the natural number set is uncountable, the set of all subsets is uncountable.

   - (ii): Let k be the size of largest finite subset. A_1: set of singleton sets, A_i: set of i-element sets, $1 \leq i \leq k$. The set A_i can be seen as $(N \times N \times \ldots \times N)$ (i times) which is a countably infinite set. The required set is $\cup_{i=1}^{k} A_i$, since countable union of countably infinite sets is countably infinite, the result is countably infinite.

   (c) The set of computational problems in computer science.
   Uncountable, for example; P_{ij}= print the open interval (i, j) where $i, j \in R$. The number of such P_{ij}'s is the cardinality of real numbers and hence uncountable.

   (d) $A = \{(x_1, \ldots, x_k) \mid x_i \in \mathtt{N}$ and k is a fixed integer $\}$
   Countably infinite as $N \times N \times \ldots \times N(k$ times) is countably infinite.
   (e) $A = \{(x_1, \ldots, x_k) \mid x_i \in \mathtt{N}$ and k is a variable integer $\}$
   Since each element (vector of k elements) is a subset of N, the number of such vectors is the cardinality of the power set of N. Therefore, uncountable.