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COM205T Discrete Structures for Computing

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Problem Session: PHP and Mathematical Induction

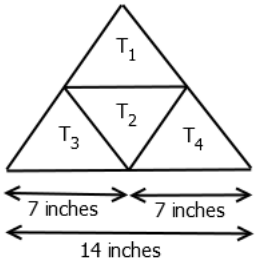
1. **King Problem** The king summoned the best mathematicians in the kingdom to the palace to find out how smart they were. The king told them "I have placed white hats on some of you and black hats on the others. You may look at, but not talk to one another. I will leave now and will come back every hour on the hour. Every time I return, I want those of who have determined that you are wearing white hats to come up and tell me immediately." As it turned out, at the n^{th} hour every one of the 'n' mathematicians who were given white hats informed the king that she knew that she was wearing a white hat? Why?

   `Solution:` We shall prove by mathematical induction on $n \geq 2$, n represents the number of mathematicians with white hats. Base: $n = 2$. There are two mathematicians with white hats and we now show that at the end of second hour mathematicians who are given white hats will inform the king about her hat's color. Note that the number of mathematicians is $k \geq 2$, out of which two are wearing white hats and the rest are wearing black hats. Let M_1, M_2 are wearing white and $M_3, \ldots, M_k$ are wearing black. Each $M_i, 3 \leq i \leq k$, sees 2 white hats and $(k-1)$ black hats. Further each M_i thinks that there are at least two 2 white hats as her hat color may be white or black. Both M_1 and M_2 can see one white hat and the rest seen are black hats. For clarity purpose, let us fix M_1. Note that king has placed some white hats (there is no scenario with only black hats). With respect to M_1, had M_1 been wearing black, M_2 would have approached the king at the end of first hour and informed her hat color. The fact that M_2 did not approach the king at the end of first hour will only imply that both M_1 and M_2 are wearing white. Subsequently, they both approach the king at the end of second hour and inform the king that they both are wearing white hats. For clarity purpose, we consider $n = 3$ case also. Let M_1, M_2, M_3 are wearing white and the rest are black. Each black hat person thinks that there are at least 3 white hats. Each of M_1, M_2, M_3 sees two white hats and the rest black hats. Had M_1 been wearing black hat, M_2 and M_3 would have approached the king at the end of second hour, that this does not happen implies that M_1 is wearing white and all three (M_1, M_2, M_3) approach the kind at the end of third hour to inform that they are wearing white hats.

   Hypothesis: Assume that there are $n = l$, $l \geq 2$, mathematicians wearing white hats and all report at the end of l^{th} hour that they are wearing white hats.
   Induction step: Consider $n = l + 1$, $l \geq 2$ mathematicians wearing white hat. Let M_{l+1} be the mathematician wearing white hat and sees l other white hats and the rest are black hats. Had M_{l+1} been white, by the induction hypothesis $M_2, \ldots, M_{l+1}$ would have approached the king at the end of l^{th} hour. Since $M_2, \ldots, M_{l+1}$ did not approach the king at the end l^{th} hour will only imply that M_{l+1} is white and all $M_1, \ldots, M_{l+1}$ approach and inform the king at the end of $(l + 1)^{th}$ hour about their hat color.
2. **Tray problem:** A tray contains labelled balls and there are finite number of balls on the tray. The game proceeds like this: if you take out a ball labelled $i \geq 2$, you can replace with

any number of balls (of course, finite number) whose labels are from $\{1, \dots, i-1\}$. There is no replacement for the ball labelled 1. The goal is to show that this game terminates, i.e. there is a sequence of replacements which will result in empty tray. We shall prove this using induction on the value of largest label.
Base: $n = 1$. Suppose the tray contains balls labelled '1' only. Clearly, there is a finite sequence of moves which will result in empty tray as there is no replacement for balls that are labelled '1'.
Hypothesis: $n = k \geq 1$. Let the largest label is k. We assume that there is a sequence of moves which will make the tray empty.
Induction step: $n = k+1$, $k \geq 1$. Let $A = \{B_{k+1} \mid$ there is a ball labelled $(k+1)$ in the tray $\}$. Since there is a replacement for B_{k+1}, start taking each ball labelled B_{k+1} from the tray till A is empty (all B_{k+1} labelled are taken out of the tray). Clearly for each pick, we will replace it with balls labelled $B_1, \dots, B_k$. Now, in the tray the highest index is k and by the induction hypothesis, there is a sequence which will make the tray empty. This completes the induction and hence the claim.

3. Five darts are thrown at an equilateral triangular target measuring 14 inches on a side. Prove that two of them must be at a distance no more than 7 inches apart.
Solution:
Divide the equilateral triangle into four T_1, T_2, T_3 and T_4 equilateral tringles as shown in *Figure*.
Pigeon holes: T_1, T_2, T_3 and T_4



Pigeons: Five darts.
PHP: At least one hole will have two darts and it will be at most 7 inches apart as distance between any two points in any T_i is at most 7. □

4. From a bin with 2 red pebbles, 5 green pebbles and 6 blue pebbles, how many must you take to be sure that you have
   - at least 2 colors ?
   - at least 3 colors ?
   - at least 2 of the same color ?
   - at least 4 of the same color ?

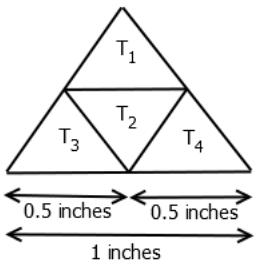
Solution:

- Seven (6+1). Note: if you pick any 7, you will always find at least 2 colors. For less than 7, some pick may be yes and the rest may be no. Since the pick is arbitrary, 7 is the right answer.

- Twelve (6+5+1).
- Four($1 \in R+\ 1 \in G+\ 1 \in B\ +1 \in \{R, G, B\}$).
- Nine ($2 \in R+\ 3 \in G+\ 3 \in B\ +1 \in \{G, B\}$).

□

5. Prove that among five points select inside an equilateral triangle with side equal to 1, there always exists a pair at a distance not greater than 0.5.
Solution:
Divide the equilateral triangle into four T_1, T_2, T_3 and T_4 equilateral tringles as shown in *Figure*.
Pigeon holes: T_1, T_2, T_3 and T_4



Pigeons: Five points.
PHP: At least one hole will have two points and it will be at most 0.5 inches apart. □

6. Suppose a postal department prints only \$5 and \$9 stamps. Prove that it is possible to make up any postage of \$$n$ using only \$5 and \$9.
Solution:
Let us prove this by induction on n. **Base Case:** $n = 35$. Seven \$5's.
Hypothesis: $n = k, k \geq 35$. Assume that \$$k$ request can be served using \$5 and \$9.
Induction Step: $n = k + 1, k \geq 35$. We will divide this into two cases
Case 1: There exist at least one \$9.
Replace one \$9 with two \$5.
Case 2: There exist at least seven \$5.
Replace seven \$5 with four \$9.
The induction is complete and hence the claim follows. Note: Induction works fine even if we assume base case to be $n = 32$. In fact any $n \geq 32$ works fine. □

7. Given any set of 7 distinct integers, there must exist 2 integers in this set whose sum or difference is divisible by 7.
Solution:
Pigeon holes: $(0, 7), (1, 6), (2, 5), (3, 4)$, 4 holes.
Pigeons: 7 distinct integers.
PHP: Place the integer x in the hole (y, z) if $x\%7 = y$ or $x\%7 = z$. Note $x\%7$ is x mod 7. There exist at least one hole with the $\lceil \frac{7}{4} \rceil = 2$ integers such that either both has the same remainder or different remainders. If it has the same remainder then, the difference is divisible by 7. If it has different remainders then, the sum is divisible by 7. □

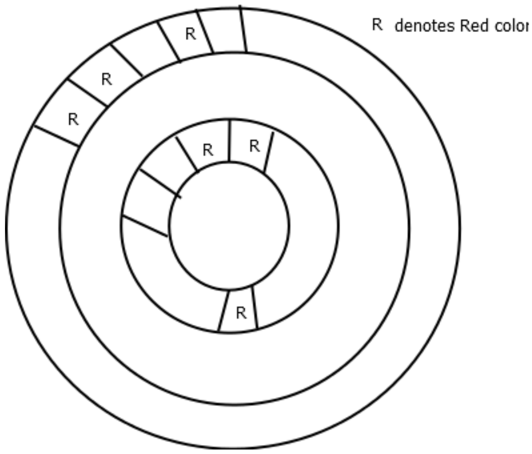
8. Among 61 integral powers of the integer 5, there are at least 6 of them that have the same remainder when divided by 12.
Solution:
Pigeon holes: $0, 1, \ldots, 11$, there are 12 holes based on possible remainders.
Pigeons: 61 different integral powers of 5.
PHP: Place the number x in the hole y if $x\%12 = y$. There exist at least one hole with

$\lceil \frac{61}{12} \rceil = 6$ numbers such that they have the same remainder when divided by 12. □

9. The circumference of two concentric disks is divided into 200 sections each. For the outer disk, 100 of the sections are painted red and 100 of the sections are painted white. For the inner disks the sections are painted red and white in an arbitrary manner. Show that it is possible to align two disks so that 100 or more of the sections in the inner disk have their colors matched with the corresponding sections on the outer disk.

Solution:

Fix the inner disk and rotate the outer in anti-clock wise direction. $O_1, \ldots, O_{200}$ denote



segments in the outer disk and $I_1, \ldots, I_{200}$ denote segments in the inner disk. If suppose O_1 is colored blue, then during anti-clock wise rotation, it will see 100 I_i's which are colored blue and hence there is a match at 100 places. Similar argument holds good if O_1 is colored red. This implies that there are 100 matches for each O_i and therefore 200×100 matches altogether for the outer disk. Since there are 200 segments in the inner disk, by pigeon hole principle, 20000 matches (alignments) are distributed among 200 segments. So the average is 100 and at least in one alignment 100 or more of the segments of outer disk will match with the segments of inner disk.

□

10. Suppose that a computer science laboratory has 15 workstations and 10 servers. A cable can be used to directly connect a workstation to a server. For each server, only one direct connection to that server can be active at any time. We want to guarantee that at any time any set of 10 or fewer workstations can simultaneously access different servers via direct connections. Although we could do this by connecting every workstation directly to every server (using 150 connections), what is the minimum number of direct connections needed to achieve this goal?

Solution:

Let $S_1, S_2, \ldots, S_{10}$ be the 10 servers and $W_1, W_2, \ldots, W_{15}$ be the 15 workstations. Connect $W_1, W_2, \ldots, W_{10}$ to $S_1, S_2, \ldots, S_{10}$, respectively (10 cables). Now, connect each workstation W_{10}, W_{11}, …, W_{15} to all 10 servers (50 cables). So that at any time any set of 10 or fewer workstations can simultaneously access different servers via direct connections and this is minimum to achieve this. Minimum number of direct connections needed to achieve this goal $= 60$. □

11. Find the least number of cables required to connect eight computers to four printers to guarantee that for every choice of four of the eight computers, these four computers can directly access four different printers.

Solution:

Let $C_1, C_2, \ldots, C_8$ be the 8 computers and P_1, P_2, P_3, P_4 be the 4 printers. Connect $C_1, C_2, \ldots, C_4$ to P_1, P_2, P_3, P_4, respectively (4 cables). Now, connect computer C_4, C_5, ..., C_8 to all 4 printers (16 cables). So that for every choice of four of the eight computers, these four computers can directly access four different printers and this is minimum to achieve this. Minimum number of direct connections needed to achieve this goal $= 20$. □