Indian Institute of Information Technology
Design and Manufacturing, Kancheepuram
Chennai 600 127, India
An Autonomous Institute under MHRD, Govt of India
An Institute of National Importance
www.iiitdm.ac.in
COM205T Discrete Structures for Computing

Instructor
N.Sadagopan
Scribe:
S.Dhanalakshmi
P.Renjith

Assignment-5 ***(Relations and Functions)***

Question 1 *Let R1 and R2 be relations on A. Prove each of the following.*

a. $r(R_1 \cup R_2) = r(R_1) \cup r(R_2)$
b. $s(R_1 \cup R_2) = s(R_1) \cup s(R_2)$
c. $t(R_1 \cup R_2) \supset t(R_1) \cup t(R_2)$
d. Show by counter example that $t(R_1 \cup R_2) \not\subset t(R_1) \cup t(R_2)$

Solution:
a. By definition $r(R) = R \cup E$. $r(R_1) = R_1 \cup E$, $r(R_2) = R_2 \cup E$.
$r(R_1) \cup r(R_2) = R_1 \cup E \cup R_2 \cup E = R_1 \cup R_2 \cup E = r(R_1 \cup R_2)$

b. By definition $s(R) = R \cup R^c$. $s(R_1) = R_1 \cup R_1^c$, $s(R_2) = R_2 \cup R_2^c$.
$s(R_1) \cup s(R_2) = R_1 \cup R_1^c \cup R_2 \cup R_2^c = R_1 \cup R_2 \cup (R_1 \cup R_2)^c = s(R_1 \cup R_2)$

c. $R_1 \subset t(R_1 \cup R_2)$. For every $(a, b), (b, c) \in R_1$, $(a, c) \in t(R_1)$. It follows that $(a, c) \in t(R_1 \cup R_2)$. Similar arguments hold for R_2. Therefore $t(R_1) \cup t(R_2) \subset t(r_1 \cup R_2)$

d. $A = \{1, 2, 3\}, R_1 = \{(1, 2)\}, R_2 = \{(2, 3)\}$
$t(R_1 \cup R_2) = \{(1, 2), (2, 3), (1, 3)\}$, $t(R_1) = \{(1, 2)\}$, $t(R_2) = \{(2, 3)\}$ and $t(R_1) \cup t(R_2) = \{(1, 2), (2, 3)\}$
here $t(R_1 \cup r_2) \not\subset t(R_1) \cup t(R_2)$

Question 2 *Show that if R is a quasi order then R is always antisymmetric.*

Solution:
Given: R is transitive and irreflexive.
For any pair $a, b \in R$, if $(a, b) \in R$ then $(b, a) \notin R$ (Suppose if $(a, b), (b, a) \in R$ then by transitivity $(a, a) \in R$, which is a contradiction to irreflexive property). Thus, R is asymmetric and hence R is antisymmetric.

Question 3 *Let* (A, R) *be a poset and B a subset of A. Prove the following*

a. If b is a greatest element of B, then b is a maximal element of B
b. If b is a greatest element of B, then b is lub of B

Solution:
a. An element $b \in B$ is a *greatest element* of B if for every $b' \in B, b' \preceq b$. An element $b \in B$ is

a maximal element of B if $b \in B$ and there does not exist $b' \in B$ such that $b \neq b'$ and $b \preceq b'$. Therefore if b is a greatest element, then there does not exist $b' \in B$ such that $b \neq b'$ and $b \preceq b'$, implies that b is a maximal element.

b. An element $b \in A$ is upper bound for B if for every element $b' \in B$, $b' \preceq b$. An element $b \in A$ is a least upper bound (lub) for B if b is an upper bound and for every upper bound b' of B, $b \preceq b'$. Therefore, if b is a greatest element, then b is clearly an upper bound. Since $b \in B$, it must be the case that $b \preceq b'$ for every upper bound b'. Therefore, b is lub.

Question 4 *Construct examples of the following sets:*
a) A non-empty linearly ordered set in which some subsets do not have a least element.
b) A non-empty partially ordered set which is not linearly ordered and in which some subsets do not have a greatest element. Construct both finite and infinite examples.
c) A partially ordered set with a subset for which there exists a glb but which does not have a least element. Construct both finite and infinite examples.
d) A partially ordered set with a subset for which there exists an upper bound but not a least upper bound. Construct both finite and infinite examples.

Solution:
(a) $(I, \leq)$
(b) **Example: Finite set** The poset given in Figure 1 is not a linearly ordered set and the

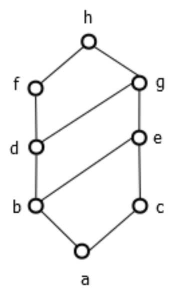


Fig. 1.

subset $\{d, e\}$ does not have the greatest element.

Example: Infinite Set $(N\backslash\{0\}, |)$, where $a \mid b$ denotes a divides b. The set itself does not have the greatest element.

(c) **Example: Finite set** The poset given in Figure 1, has a subset $\{d, e\}$ for which there exists a glb, $\{b\}$, but which does not have a least element.

Example: Infinite Set $(N\backslash\{0\}, |)$, where $a \mid b$ denotes a divides b. The subset $\{4, 6\}$ has a glb, $\{2\}$, but does not have a least element.
(d) **Example: Finite set** The poset given in Figure 2, has a subset $\{a\}$ for which there exists a upper bound, $\{b, c, d, e, f, g\}$ but no least upper bound.

Example: Infinite set Set: R, Subset: $(0, 1)$, Relation: less than. Upper bound $\{x \mid x \geq 1\}$, however the subset has no least upper bound.

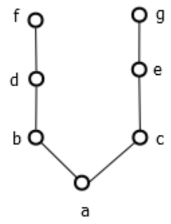


Fig. 2.

Question 5 *Construct a bijection from A to B*

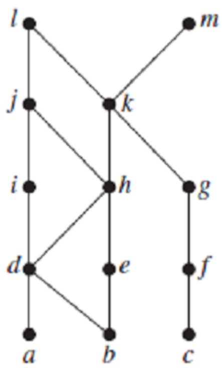
a. $A = I, B = N$
b. $A = N, B = N \times N$
c. $A = [0, 1), B = (\frac{1}{4}, \frac{1}{2}]$
d. $A = R, B = (0, \infty)$

Solution:
(a). $f(x) = 2|x|$ if $x \geq 0$
$f(x) = 2|x| + 1$ if $x < 0$

(b). Construct $N \times N$ matrix and enumerate in a systematic way,
i.e. $(0, 0), (1, 1), (1, 2), (2, 1), (3, 1), (2, 2), (1, 3), (1, 4), (2, 3), ...$ This shows that there is one-one correspondence between every element of N and an element of $N \times N$ matrix.
(c). $f(x) = \frac{2-x}{4}$

(d). $f(x) = e^x$

Question 6 *For the following hasse diagram, find*



(a) Find the maximal elements.
Solution: $\{l, m\}$

b) Find the minimal elements.
Solution: $\{a, b, c\}$

c) Is there a greatest element?

Solution: No

d) Is there a least element?
Solution: No

e) Find all upper bounds of $\{a, b, c\}$.
Solution: $\{k, l, m\}$

f) Find the least upper bound of $\{a, b, c\}$, if it exists.
Solution: $\{k\}$

g) Find all lower bounds of $\{f, g, h\}$.
Solution: NIL

h) Find the greatest lower bound of $\{f, g, h\}$, if it exists.
Solution: NIL

Question 7 *Using PIE (principle of inclusion and exclusion), Find the number of positive integers not exceeding* 100 *that are either odd or the square of an integer*

Solution:
Number of odd numbers $= |O| = 50$
Number of square numbers $= |S| = 10$
Number of odd square numbers $= |O \cap S| = 5$
$|O \cup S| = |O| + |S| - |O \cap S|$
$= 50 + 10 - 5 = 55$

Question 8 *Using PIE, How many bit strings (binary) of length eight do not contain six consecutive 0's.*

Solution:
Number of bit strings of length 8 do not contain six consecutive 0's = Total number of bit strings of length 8 - Number of bit strings containing 6 consecutive 0's.
Note that the total number of bit strings of length 8 $= 2^8 = 256$
Number of bit strings containing 6 consecutive 0's
Let k denote the substring with six zeroes and a, b are the other two bits.
Number of bit strings containing 6 consecutive 0's = number of bit strings of length 8 of the form $kab +$ number of bit strings of length 8 of the form $abk +$ number of bit strings of length 8 of the form akb **-** number of bit strings of length 8 of the form kab and abk **-** number of bit strings of length 8 of the form abk and akb **-** number of bit strings of length 8 of the form kab and $akb +$ number of bit strings of length 8 of the form kab, abk and akb.
$= 4 + 4 + 4 - 1 - 2 - 2 + 1 = 8$
Therefore, number of bit strings of length 8 do not contain six consecutive 0's = 256-8 = 248.

Question 9 *Using PIE, count the number of primes between* 2 *and* 100

Solution:
Consider the prime factors $2, 3, 5, 7$. P_i represents number of elements in the range 2-100 that are divisible by i.
$|P_i P_k \ldots P_j|$ represents the number of elements in the range 2-100 that are divisible by $i \times k \times \ldots \times j$
Number of primes between 2 and 100 = 99 - number of numbers that are multiples of 2,3,5,7 + 4 (the numbers 2,3,5,7).
Number of prime numbers (excluding 2,3,5,7)= 99 - $|P_2|$-$|P_3|$-$|P_5|$-$|P_7|$
+$|P_2P_3|$+$|P_3P_5|$+$|P_5P_7|$+$|P_2P_5|$+$|P_2P_7|$+$|P_3P_7|$-$|P_2P_3P_5|$-$|P_2P_3P_7|$-$|P_3P_5P_7|$
-$|P_2P_5P_7|$+$|P_2P_3P_5P_7|$
$= 99 - 50 - 33 - 20 - 14 + 16 + 10 + 7 + 6 + 4 + 2 - 3 - 2 - 1 - 0 + 0 = 21$
Therefore, the total number of primes in the range 2-100 = 21 + 4 =25.

Question 10 *Using PIE, the number of solutions to* $x_1 + x_2 + x_3 = 10$ *with* $x_1 \leq 2, x_2 \leq 2, x_3 \leq 3$.

Solution:
Number of solutions = Number of solutions without any constraints - Number of solutions with $(x_1 \geq 3 \vee x_2 \geq 3 \vee x_3 \geq 4)$
Generic Approach: Let $x_1 + x_2 + x_3 = r$ such that $x_i \geq 0$. The number of solutions to this equation is the number of ways distributing r balls into 3 boxes, which is equivalent to introducing two 0's into r-bit string consisting of all 1's. In other words, the number of permutations (reorderings) of a string containing r ones and 2 zeros. Let $x_1 + x_2 + x_3 = r$ such that $x_i \geq 1$. The number of solutions to this equivalent to the number of solutions to $y_1 + y_2 + y_3 = r - 3$ such that $y_i \geq 0$. Using this approach we shall now do the counting.

Number of solutions without any constraints = Number of reorderings of 10 ones and 2 zeroes = $12C_2$ = 66
Let A denotes $x_1 \geq 3$, B denotes $x_2 \geq 3$ and C denotes $x_3 \geq 4$.
Number of solutions with $(x_1 \geq 3 \vee x_2 \geq 3 \vee x_3 \geq 4)$ = $n(A \cup B \cup C)$ = $n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
$n(A) = n(B)$ = Reordering 7 one's and 2 zeroes (as three one's are already fixed) = $9C_2$ = 36
$n(C)$ = Reordering 6 one's and 2 zeroes (as four one's are already fixed) = $8C_2$ = 28
$n(A \cap B)$ = Reordering 4 one's and 2 zeroes (as six one's are already fixed) = $6C_2$ = 15
$n(A \cap C) = n(B \cap C)$ = Reordering 3 one's and 2 zeroes (as seven one's are already fixed) = $5C_2$ = 10
$n(A \cap B \cap C)$ = Reordering zero one's and 2 zeroes (as all the 10 one's are fixed) = $2C_2$ = 1
Number of solutions with $(x_1 \geq 3 \vee x_2 \geq 3 \vee x_3 \geq 4)$ = 36+36+28-15-10-10+1=66
Therefore, the number of solutions = 66-66 =0.