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COM205T Discrete Structures for Computing

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Assignment 4.5 - Relations

Question 1 *Let $A = \{1, 2\}$. Construct the set $\rho(A) \times A$, where $\rho(A)$ is the power set (set of all subsets) of A.*

Solution:
$A = \{1, 2\}$; $\rho(A) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$
$\rho(A) \times A = \{(\phi, 1), (\phi, 2), (\{1\}, 1), (\{1\}, 2), (\{2\}, 1), (\{2\}, 2), (\{1, 2\}, 1), (\{1, 2\}, 2)\}$

Question 2 *Given that $A \subseteq C$ and $B \subseteq D$, show that $A \times B \subseteq C \times D$.*

Solution:
To show that $A \times B \subseteq C \times D$, consider any arbitrary pair $(a, b) \in A \times B$, where $a \in A, b \in B$.
$A \subseteq C \Rightarrow a \in C$ *and* $B \subseteq D \Rightarrow b \in D$. Thus, $(a, b) \in C \times D$.
It follows that $A \times B \subseteq C \times D$.

Question 3 *Given that $A \times B \subseteq C \times D$, does it necessarily follow that $A \subseteq C$ and $B \subseteq D$?*

Solution:
It is not necessary that if $A \times B \subseteq C \times D$ then, $A \subseteq C$ and $B \subseteq D$.
Counter example:
Let $A = \{1, 2\}, B = \phi$, $C = \{3\}$ and $D = \{4\}$
$A \times B = \phi, C \times D = \{(3, 4)\}$
Clearly, $A \times B \subseteq C \times D$ but $A \nsubseteq C$

Question 4 *Is it possible that $A \subseteq A \times A$ for some set A ?*

Solution:
Yes. If $A = \phi$ then $A \subseteq A \times A$.

Question 5 *For each of the following check whether 'R' is Reflexive, Symmetric, Anti-symmetric, Transitive, an equivalence relation, a partial order.*

1. $R = \{(a, b) \mid a - b$ *is an odd positive integer* $\}$.
2. $R = \{(a, b) \mid a = b^2$ *where* $a, b \in I^+\}$.
3. *Let P be the set of all people. Let R be a binary relation on P such that (a, b) is in R if a is a brother of b.*
4. *Let R be a binary relation on the set of all strings of $0'$s and $1'$s, such that* $R = \{(a, b) \mid a$ *and* b *are strings that have same number of* $0's\}$.

Solution:

Q.No	Reflexive	Symmetric	Anti-symmetric	Transitive	Equivalence	Poset
1.	×	×	✓	×	×	×
2.	×	×	✓	×	×	×
3.	×	×	×	✓	×	×
4.	✓	✓	×	✓	✓	×

Question 6 *Let R be a symmetric and transitive relation on set A. Show that if for every 'a' in A there exists 'b' in A, such that (a, b) is in R, then R is an equivalence relation.*

Solution:
Given: $\forall a \; \exists b \; (b \in A \wedge (a, b) \in R)$.
To prove: R is reflexive

Since R is symmetric, if $(a, b) \in R \Rightarrow (b, a) \in R$ and since R is transitive, $(a, b) \in R, (b, a) \in R \Rightarrow (a, a) \in R$ and this argument is true $\forall a \in A$. Therefore, R is reflexive. Hence R is an equivalence relation.

Question 7 *Let R be a transitive and reflexive relation on A. Let T be a relation on A, such that (a, b) is in T if and only if both (a, b) and (b, a) are in R. Show that T is an equivalence relation.*

Solution:
To prove that T is equivalence relation we need to prove T is reflexive, T is symmetric and T is transitive.
Given that $(a, b) \in T$ iff $(a, b), (b, a) \in R$
Clearly $(a, a) \in T, \forall a \in A$, This is true because R is transtive (reflexive). This proves that T is reflexive.
If $(a, b) \in T$ we need to prove that $(b, a) \in T$. By the hypothesis (given condition), it is easy to see that $(b, a) \in T$. Hence T is symmetric.
If $(a, b) \in T$ and $(b, c) \in T$, we need to prove that $(a, c) \in T$.
$(a, b) \in T \rightarrow (a, b), (b, a) \in R$
$(b, c) \in T \rightarrow (b, c), (c, b) \in R$
Since R is transitive $(a, c) \in R$ and $(c, a) \in R$, this implies that $(a, c) \in T$. Hence T is transitive.
Therefore, T is an equivalence relation.

Question 8 *Let R be a binary relation. Let $S = \{(a, b) \mid (a, c) \in R$ and $(c, b) \in R$ for some $c\}$. Show that if R is an equivalence relation, then S is also an equivalence relation.*

Solution:
To Prove: S is reflexive.
Since R is reflexive $(a, a) \in R \; \forall a \in A$. Clearly $(a, a) \in S \; \forall a \in A$. This proves that S is reflexive.
To prove:S is symmetric
$(a, b) \in S \rightarrow \exists x \; (a, x) \in R, (x, b) \in R$
Since R is symmetric $(x, a) \in R, (b, x) \in R$.
Therefore by given definition, $(b, a) \in S$.
This proves that S is symmetric.
To prove: S is transitive
If $(a, b) \in S$ and $(b, c) \in S$ we need to prove that $(a, c) \in S$.
$(a, b) \in S \rightarrow \exists d \; (a, d), (d, b) \in R$
R is symmetric $\rightarrow (d, a), (b, d) \in R$
$\Rightarrow (a, b) \in R, (b, a) \in R$
$(b, c) \in S \rightarrow \; \exists e \; (b, e), (e, c) \in R$
R is symmetric $\Rightarrow (e, b), (c, e) \in R$
$\Rightarrow (b, c) \in R, (c, b) \in R$
Since R is transitive, $(a, c) \in R, (c, a) \in R$ ——(1)
Since R is reflexive, $(c, c) \in R$ ——(2)

From (1) and (2) it follows that $(a, c) \in S$

Therefore, S is transitive and hence an equivalence relation.

Question 9 *Let R be a reflexive relation on a set A. Show that R is an equivalence relation if and only if (a, b) and (a, c) are in R implies that (b, c) is in R.*

Solution:
Necessity: Given that R is an equivalence relation, we need to prove that $(a, b), (a, c) \in R \rightarrow (b, c) \in R$
Since R is symmetric, $(a, b) \in R \Rightarrow (b, a) \in R$
Since R is transitive, $(b, a), (a, c) \in R \Rightarrow (b, c) \in R$
Hence necessity is proved.
Sufficiency: To show that R is an equivalence relation, we need to show that R is symmetric and transitive.
By definition, $(a, b), (a, c) \in R \Rightarrow (b, c) \in R$
Also $(a, c), (a, b) \in R \Rightarrow (c, b) \in R$
Therefore, R is symmetric.
To prove transitivity, if $(x, y), (y, z) \in R$ then $(x, z) \in R$
$(x, y) \in R,\ (a, x) \& (a, y) \in R$
$(y, z) \in R,\ (a, y) \& (a, z) \in R$
$(a, x) \& (a, z) \in R \Rightarrow (x, z) \in R$. Hence R is transitive.
Therefore R is an equivalence relation. Hence sufficiency is proved.

Question 10 *Let A be a set with n elements. Using mathematical induction,*

1. *Prove that there are 2^n unary relations on A.*
2. *Prove that there are 2^{n^2} binary relations on A.*
3. *How many ternary relations are there on A ?*

Solution:

1. Let us prove this by induction on the number of elements in A, n.
   Base Case: If $n = 0$ then, the number of relations is $2^0 = 1$ (Empty set). If $n = 1$ then, the number of unary relations is $2 = 2^{1^2}$ (If $A = \{x\}$ then, the unary relations on $A = \{\phi, x\}$)
   Hypothesis: Assume that the statement is true for $n = k, k \geq 1$
   Induction Step: Let A be the set with $n = k + 1$ elements, $k \geq 1$.
   The number of unary relations on a set with $k + 1$ elements $=$ Number of unary relations on a set with k elements $+\ 2^k$ ($\because (k+1)^{th}$ element can be placed in each of 2^k subsets of k elements) $= 2^k + 2^k = 2^{k+1}$.
2. Let us prove this by induction on the number of elements in A, n.
   Base Case: If $n = 0$ then, the number of relations is $2^0 = 1$ (Empty set). If $n = 1$ then, the number of binary relations is $2 = 2^{1^2}$ (If $A = \{x\}$ then, $A \times A = \{\phi, (x, x)\}$).
   Hypothesis: Assume that the statement is true for $n = k, k \geq 1$
   Induction Step: Let A be the set with $n = k+1$ elements, $k \geq 1$. Let $A = \{x_1, x_2, \ldots, x_k, x_{k+1}\}$
   For k elements, the number of binary relations are 2^{k^2}. For $(k + 1)^{th}$ element, we have the following $2k + 1$ binary elements:
   (x_1, x_{k+1}), (x_2, x_{k+1}), ..., (x_k, x_{k+1}), (x_{k+1}, x_1), (x_{k+1}, x_2), ..., (x_{k+1}, x_k), (x_{k+1}, x_{k+1}).
   Therefore, number of binary relations for the set $A = 2^{k^2}\ .\ 2^{2k+1} = 2^{k^2+2k+1} = 2^{(k+1)^2}$.

3. Number of ternary relations on $A = 2^{n^3}$