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COM205T Discrete Structures for Computing

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Assignment-4 Solution *(Pigeonhole Principle)*

Question 1: Given 8 distinct integers $(x_1, x_2, \ldots, x_8)$, show that there exist a pair with the same remainder when divided by 7.
Label the pigeonholes with possible remainders when a number is divided by 7. i.e., labels are $0, 1, \ldots, 6$. Thus there are 7 pigeonholes, and given that 8 distinct integers, by pigeonhole principle, there exist at least one label (pigeon hole) having more than one integer. This implies that there exist a pair of integers with same remainder when divided by 7.

Question 2: Given 7 distinct integers, there must exist two integers such that the sum or difference is divisible by 6.
Like before, pigeon holes denote possible remainders when a number is divided by 6. Due to 7 distinct integers, by pigeonhole principle, there exist at least one hole having more than one integer, say a, b. It follows that difference of a and b is a multiple of 6. If both a and b leave remainder '3' then sum or difference is divisible by 6.

Question 3: Given $n + 1$ distinct integers, then there is some pair of them such that their difference is divisible by the positive integer n.
Each pigeonhole groups integers having same remainder when divided by n. Thus there are n pigeonholes and $n+1$ distinct integers. By pigeonhole principle, there exist at least one remainder class having more than one integer, say a, b. It follows that $a = n.x + r$ and $b = n.y + r$. Without loss of generality, let $a > b$. This implies that $x > y$ and $a - b = n(x - y)$. Therefore, difference of a and b is a multiple of n.

Question 4: Given 37 distinct positive integers, then there must be at least 4 of them that have the same remainder when divided by 12.
Consider the pigeonholes to be the class of integers having same remainder when divided by 12. There exist $37 = 12 \times 3 + 1$ distinct positive integers (*pigeons*) and 12 remainder classes (*pigeonholes*). Therefore by generalized pigeonhole principle, there exist a remainder class having at least $3+1 = 4$ integers. Therefore there exist at least 4 distinct integers with same remainder.

Question 5: Show that one of any n consecutive integers is divisible by n.
On the contrary, we assume that there does not exist a number divisible by n in a set of n consecutive integers. Place integers in its congruence classes, $i \ mod \ n$, $1 \leq i \leq n - 1$. Each congruence class is a pigeonhole, observe that n integers (*pigeons*) are distributed among $n - 1$ pigeon holes and by PHP, there is a hole with more than one integer. This is a contradiction to the fact that the n integers which are consecutive and distinct have n distinct remainders. Therefore our assumption is wrong and one of any n-consecutive integers is divisible by n.

Question 6: A student has 37 days to prepare for an examination. From past experience she knows that she will require no more than 60 hours of study. She also knows that she wishes to

study at least 1 hour per day. Show that no matter how she schedules her study time (a whole number of hours per day however) there is a succession of days during which she would have studied exactly 13 hours.
Let s_i, $1 \leq i \leq 37$ be the number of hours studied till i^{th} day. Then,
$s_1 < s_2 < \ldots < s_{37} \leq 60$.
$s_1 + 13 < s_2 + 13 < \ldots < s_{37} + 13 \leq 60 + 13 = 73$.
Note that for all $1 \leq i, j \leq 37$, $s_i \neq s_j$ where $i \neq j$. There exist $2 \times 37 = 74$ summands (*pigeons*), and 73 distinct integer values (*pigeonholes*). Therefore, there exist two summands having same value. i.e., $s_i = s_j + 13$ and this implies that $s_i - s_j = 13$. Thus there exist a period of $i - j$ consecutive days (day $j + 1, \ldots, i$) in which she spent 13 hours for studying.

Question 7: Given n pigeons to be distributed among k pigeonholes:
What is a necessary and sufficient condition on n and k that, in every distribution, at least two pigeonholes must contain the same number of pigeons.
Consider the scenario in which pigeonhole P_i is filled with i pigeons, $0 \leq i \leq k-1$. If P_k is filled with any value between 0 and $k-1$, then there are two pigeon holes containing the same number of pigeons. Moreover, if P_k contains any number greater than $k-1$ (k or more) then our claim need not be true always. Therefore, the number of pigeons $n = x + y$, where $x = (0 + 1 + \ldots + k - 1)$ and $y \leq k - 1$. i.e. $n \leq \frac{(k-1)k}{2} + k - 1$.

Question 8: What is the value of n (minimum n) such that in any group of n people you see either 3 mutual enemies or 4 mutual friends.
Minimum value of n for which in any group of n people there exist either 3 mutual enemies or 4 mutual friends is 9. Consider 9 persons $p_1, p_2, \ldots, p_9$ and note that

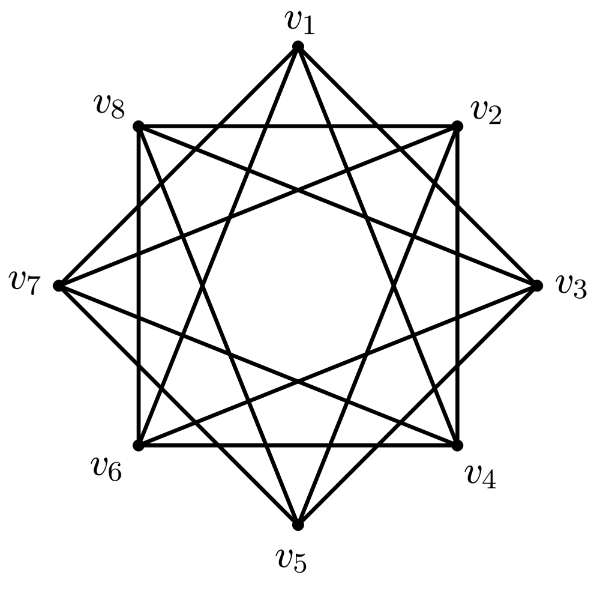

Fig. 1. Fig: An illustration on $n = 8$

In the above figure, node v_i, $1 \leq i \leq 8$ represents people and edge between node represent friend relation between them. The figure is a counter example illustrating that in a group of 8 people, there need not be 3 mutual enemies or 4 mutual friends. Now we prove that there exist either 3 mutual enemies or 4 mutual friends among a group of 9 people. Consider 9 nodes

representing people with every pair of vertices connected by an edge representing relationships. The edge is colored *Blue* for friend and *Red* for enemy relationships. We now show that their exist a red triangle (representing 3 mutual enemies) or blue K_4 (4 vertices with 6 edges among them) representing 4 mutual friends. Note $N_B(x) = \{y \mid (x, y) \text{ } is \text{ } blue\}$ and $N_R(x) = \{y \mid (x, y) \text{ } is \text{ } red\}$
Case 1: If there exist a vertex v_i with at least 6 blue edges incident on it, then in $N_B(v_i)$, there exist either a red triangle induced on $R \subseteq N_B(v_i)$ or blue triangle B' induced on $B \subseteq N_B(v_i)$ as $|N_B(v_i)| = 6$. Therefore there exist a red triangle induced on R or a blue K_4 induced on $B' \cup v_i$ in the graph.
Case 2: If there exist a vertex v_i with at least 4 red edges incident on it, then we can see the following. If there exist $v_j, v_k \in N_R(v_i)$, such that (v_j, v_k) is red, then $\{v_i, v_j, v_k\}$ induces a red triangle. On the other hand, if there does not exist $v_j, v_k \in N_R(v_i)$, such that (v_j, v_k) is red, then $\{v_i\} \cup N_R(v_i)$ has an induced K_4.
Case 3: All the vertices v_i in the graph are having at most 3 red edges and at most 5 blue edges incident on them. Since all edges are colored either blue or red, it follows that in the graph all vertices have exactly 3 red edges and 5 blue edges incident on them. It follows that there are $(9 \times 3)/2 = 13.5$ red edges, similarly, there are $(9 \times 5)/2 = 22.5$ blue edges. Note that sum of the degrees in a graph is twice the number of edges, which implies that the number of edges is always even. Further, the set of edges can be partitioned into 'red' and 'blue' edges. In our case, degree sums due to 'red' edges and 'blue' edges yield a non-integer, a contradiction. Therefore, this case does not occur. Among the mutually exclusive and exhaustive cases discussed, case $1, 2$ shows that there exist either a red triangle or a blue K_4 and in case 3 such a graph does not exist. Therefore there exist 3 mutual enemies (red triangle) or 4 mutual friends (blue K_4) in a group of at least 9 people (vertices).