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COM205T Discrete Structures for Computing

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Assignment 3 - Proof Techniques

Direct Proof/Proof by contradiction.

Question 1 *If P is a prime > 3, then P^2 has the form $12k + 1$, where k is an integer.*

Solution 1:
Since P is a prime number, P is an odd number. This implies, P^2 is an odd number. Thus, $P^2 - 1$ (even number) is divisible by 4 ($\because$ $P^2 - 1$ can be written as $(P + 1)(P - 1)$, where $P + 1$ is even and $P - 1$ is even). Also, $P^2 - 1$ is divisible by 3 ($\because$ for any three consecutive integers $P - 1, P, P + 1$, any one integer is divisible by 3. But, P is a prime number and > 3. Therefore, either $P - 1$ or $P + 1$ is divisible by 3). We know that, Let $a \mid b$ and $c \mid b$. If $gcd(a, c) = 1$ then, $ac \mid b$. Since $gcd(4, 3) = 1$, 12 divides $P^2 - 1$ i.e., $P^2 - 1 = 12k$, where k is any integer. Thus, $P^2 = 12k + 1$.

Solution 2:
Any prime number > 3 can be written in the form $6m \pm 1$, where m is a positive integer. Thus, $P^2 = 36m^2 \pm 12m + 1 = 12(3m^2 \pm m) + 1 = 12k + 1$, where $k = 3m^2 \pm m$.

Question 2 *If an integer is simultaneously a square and a cube (ex: $64 = 8^2 = 4^3$), verify that the integer must be of the form $7n$ or $7n + 1$.*

Solution:
Direct proof: Let $z = x^2$ and $z = y^3$ for some $x, y \in \mathbb{I}$. Note that any number $x \in \mathbb{I}$ can be represented as $x \ mod \ 7 = i$, $0 \leq i \leq 6$. This implies $x^2 \ mod \ 7 = j$, $j = \{0, 1, 2, 4\}$. Similarly, $y \ mod \ 7 = i$, $0 \leq i \leq 6$ implies that $y^3 \ mod \ 7 = k$, $k = \{0, 1, 6\}$. It follows that if $z \ mod \ 7 = 0$ or $z \ mod \ 7 = 1$. Therefore, $z = 7n$ or $z = 7n + 1$.

Question 3 *The circumference of a 'roulette wheel' is divided into 36 sectors to which the numbers $1, 2, \ldots, 36$ are assigned in some arbitrary manner. Show that there are 3 consecutive sectors such that the sum of their assigned numbers is at least 56.*

Solution:
Let a_i denotes the sum of three consecutive sectors from sector i, $1 \leq i \leq 36$. On the contrary, assume that the sum of any three consecutive sectors is ≤ 55. Therefore,

$$\begin{aligned}
\sum_{i=1}^{36} a_i &\leq 36 \times 55 \\
3 \times (1 + 2 + \ldots + 36) &\leq (36 \times 55) \\
3 \times \frac{36 \times 37}{2} &\leq (36 \times 55) \\
111 &\leq 110 \quad (Which \ is \ a \ contradiction)
\end{aligned}$$

Thus, our assumption that the sum of any three consecutive sectors is ≤ 55 is wrong. Therefore, there exist 3 consecutive sectors such that the sum of their assigned numbers is at least 56.

Question 4 *If there are 104 different pairs of people who know each other at a party of 30 people, then show that some person has 6 or fewer acquaintances.*

Solution:
On the contrary assume that all persons are having at least 7 acquaintances. Therefore, the number of distinct acquaintance pair is at least $30 \times 7/2 = 105$. This is contradiction to the fact that there are 104 different pair of acquaintances. Therefore, our assumption is wrong and it follows that there exist at least a person with 6 or fewer acquaintances.

Mathematical Induction

Question 5 *Prove by induction : For $n \geq 1$, $8^n - 3^n$ is divisible by* 5.

Solution:
Base case: $n = 1$, $8^1 - 3^1$ is divisible by 5. *Induction hypothesis:* Assume that $8^n - 3^n$ is divisible by 5 for all $n \geq 1$. *Induction step:* For $n \geq 1$, consider $8^{n+1} - 3^{n+1} = (8.8^n - 3.3^n) = (5+3)8^n - 3.3^n = 5.8^n + 3(8^n - 3^n)$. By the induction hypothesis, $8^n - 3^n$ is divisible by 5 and hence, $= 5.8^n + 3(8^n - 3^n)$ is divisible by 5. Therefore, we can conclude that $8^n - 3^n$ is divisible by 5 for all $n \geq 1$.

Question 6 *Prove by induction: a number, given its decimal representation is divisible by 3 iff the sum of its digits is divisible by three.*

Solution:
Let us prove this by induction on number of digits, n.
Base Case: $n = 1$, clearly, then a number is divisible by 3 iff the sum of the digits is divisible by 3. For example, the numbers 3,6,9 satisfy this case.
Hypothesis: Assume that the statement holds for $n = k$, $k \geq 1$. i.e., a number composed of k digits is divisible by 3 iff the sum of its digits is divisible by 3.
Induction Step: Let $n = k + 1$, $k \geq 1$. Let x be a number composed of $k + 1$ digits. Our claim is to prove that x is divisible by 3 iff the sum of the digits in x is divisible by 3.

Let the decimal expansion of $x = a_k a_{k-1} \ldots a_0$ be,
$x = a_k 10^k + a_{k-1}10^{k-1} + \ldots + a_0$. $- - - - - - - - - - > (1)$

Since, $a_m 10^m$ can be written as $a_m + a_m(10^m - 1)$, equation (1) can be written as follows:

$$x = (a_k + a_{k-1} + \ldots + a_0) + (a_k(10^k - 1) + a_{k-1}(10^{k-1} - 1) + \ldots + a_1(10^1 - 1)),$$

Since, 3 divides $10^n - 1$, implies that 3 divides $(a_k(10^k - 1) + a_{k-1}(10^{k-1} - 1) + \ldots + a_1(10^1 - 1))$. By the hypothesis, $(a_{k-1} + \ldots + a_0)$ is divisible by 3 iff $a_{k-1} \ldots a_0$. Thus, x is divisible by 3 iff a_k is divisible by 3. i.e., x is divisible by 3 iff $(a_k + a_{k-1} + \ldots + a_0)$ is divisible by 3. Therfore, x is divisible by 3 iff the sum of the digits in x is divisible by 3.

Question 7 *For each positive integer n, there are more than n prime integers.*

Solution:
Base case: $n = 1$. $\{2, 3, \ldots\}$ are prime integers. Clearly, for the integer '1', there exist more than one.
Induction hypothesis: Assume for $n = k, k \geq 1$, that there exist more than k prime integers. Let the prime numbers be $p_1, p_2, \ldots, p_k, p_{k+1} \ldots$.
Induction step: We claim that for $n = k + 1$, $k \geq 1$ there exist more than $k + 1$ prime numbers. Consider the number $P = p_1 \cdot p_2 \ldots p_k \cdot p_{k+1} + 1$, i.e. P is one plus the product of the prime numbers $p_1, p_2, \ldots, p_{k+1}$.
We consider the following cases to complete the proof.
Case a: If P is a prime number, then there exist more than $k+1$ prime numbers with $(k+2)^{nd}$ prime number being P.
i.e., $\{p_1, p_2, \ldots, p_k, p_{k+1}, P\}$ are the set of $(k + 2)$ prime numbers.
Case b: If P is not a prime number, then note that there exist a prime factorization for P and none of $\{p_1, p_2, \ldots, p_k, p_{k+1}\}$ are its prime factors. This implies that there exist a prime factor p_{k+2} for P such that $p_{k+2} \neq p_i$, $1 \leq i \leq k + 1$. Therefore, $\{p_1, p_2, \ldots, p_k, p_{k+1}, p_{k+2}\}$ are prime numbers with cardinality more than $k+1$. The induction is complete and hence the claim follows.

Question 8 *Show that any integer composed of 3^n identical digits is divisible by 3^n. (for example: 222 is div by 3, 555,555,555 is div by 9)*

Solution:
We shall prove this by induction on n.
Base Case: For $n = 1$, we note that any 3-digit integer with 3 identical digits is divisible by 3. Since, for any $k \in \{1, \ldots, 9\}$, $kkk = k \cdot (111)$. Further, 111 is divisible by 3. Therefore, kkk is divisible by 3.
Hypothesis: Assume that the statement is true for $n = k$, $k \geq 1$.
Induction Step: For $n = k + 1, k \geq 1$. Let x be an integer composed of 3^{k+1} identical digits. We note that x can be written as
$x = y \times z$
where y is an integer composed of 3^k identical digits, and $z = 10^{2 \cdot 3^k} + 10^{3^k} + 1$.
For example, $x = 666666666 = 666 \times 1001001 = y \times (10^{2 \cdot 3^1} + 10^{3^1} + 1)$. y is divisible by 3^k by the hypothesis and z is divisible by 3 (sum of the digits is divisible by 3). Thus x is divisible by 3^{k+1}.

Pigeon Hole Principle

Question 9 *A person takes at least one tablet a day for 50 days. He takes 90 tablets altogether. Is it true that during some sequence of consecutive days he has taken exactly 24 tablets. Justify your answer.*

Solution:
Let a_i be the number of tablets the patient has taken till the end of the i^{th} day. Thus we have the following sequence:
$1 \leq a_1 < a_2 < \ldots < a_{50} = 90$.
Thus we have
$1 + 24 \leq a_1 + 24 < a_2 + 24 < \ldots < a_{50} + 24 = 90 + 24$.
i.e.,
$25 \leq a_1 + 24 \leq a_2 + 24 \leq \ldots \leq a_{50} + 24 = 114$.

Thus, among all the numbers: $a_1, a_2, \ldots, a_{50}, a_1 + 24, \ldots, a_{50} + 24$ are 100 numbers (pigeons) from 114 (pigeon holes). So, there is no possibility of two numbers to be equal (Since a patient takes at least one tablet a day). Thus, there is no sequence of consecutive days where the patient has taken exactly 24 tablets.

Question 10 *Show that one of any n-consecutive integers is divisible by n.*

Solution:
On the contrary, we assume that there does not exist a number divisible by n in a set of n consecutive integers. We can place integer i in congruence class j, where $j = i \ mod \ n$, $1 \leq j \leq n-1$ corresponding to pigeon holes. Observe that n integers (pigeons) are there and by pigeonhole principle, there exist a class with more than one integer, say a, b where $a = x.n + r$ and $b = y.n + r$. Note that x and y differ by at least one and it follows that there exist at least $n + 1$ consecutive integers from a to b inclusive of both. This is a contradiction to the fact that there are n consecutive integers. Therefore our assumption is wrong and one of any n-consecutive integers is divisible by n.

Question 11 *Show that among $(n + 1)$ positive integers less than or equal to $2n$, there are 2 consecutive integers.*

Solution:
Pigeon holes: $(1, 2), (3, 4), \ldots, (2n - 1, 2n)$, n pigeon holes
Pigeons: $n+1$ pigeons. Choose $n+1$ distinct numbers from the $2n$ positive integers. Place pigeon x in the hole (a, b) if $a = x$ or $b = x$.
PHP: At least 2 pigeons will be placed in a hole and since $(n+1)$ integers are distinct those two pigeons are consecutive numbers (by the definition of pigeon holes).

Question 12 *Show that in a group of five people (where any two people are either friends or enemies), there are not necessarily three mutual friends or three mutual enemies.*

Solution:
Consider a person A and divide the remaining 4 persons into two sets, friends and the enemies of A. There exist at least two persons the friends set or in the enemies set of A by pigeonhole principle. We can see the below possibilities

Cardinality of friend set of A	**Cardinality of enemy set of A**
2	2
1	3
3	1
0	4
4	0

Consider the possibility where there exist 2 friends B, E and 2 enemies C, D of A. If B and E are friends, then there exist three mutual friends, $\{A, B, E\}$. Therefore, we consider a scenario where B and E are enemies. Similarly, if C and D are enemies, then there exist three mutual enemies, $\{A, C, D\}$. Therefore, we consider a scenario where C and D are friends. Now if B is a friend of C, and D is a friend of E, then there does not exist three mutual friends or three mutual enemies in the scenario. Scenario in short: friend relations $(A, B), (B, C), (C, D), (D, E), (E, A)$.

Question 13 *Show that in a group of 10 people (where any two people are either friends or enemies), there are either three mutual friends or four mutual enemies, and there are either three mutual enemies or four mutual friends.*

Solution:
We shall prove there are either three mutual friends or four mutual enemies and the argument for the other claim is symmetric.
Let $A, B, \ldots, J$ be the ten persons. Take a person A: divide the remaining 9 persons into friends set of A and the enemies set of A. By PHP, $\lceil \frac{9}{2} \rceil = 5$, at least five persons either in the friends set of A or in the enemies set of A. Therefore, the possibilities are

We present case by case analysis among the above possibilities: at least 4 friends for A or

Cardinality of friend set of A	**Cardinality of enemy set of A**
5	4
6	3
7	2
8	1
9	0
4	5
3	6
2	7
1	8
0	9

at least 6 enemies for A.
Case 1: At least 4 friends for A
With respect to A, assume that (B, C, D, E) are the 4 friends. If any two of (B, C, D, E) are friends, then those two along with A forms 3 mutual friends. If none of them are friends, then (B, C, D, E) form 4 mutual enemies.
Case 2: At least 6 enemies for A
w.l.o.g. assume that (B, C, D, E, F, G) are the 6 enemies for A. W.k.t. Among 6 people, there exist either 3 mutual friends or 3 mutual enemies. If there are 3 mutual friends then there is nothing to prove or if there are three mutual enemies then this 3 along with A form four mutual enemies.

Question 14 *Show that if $n+1$ integers are chosen from the set $\{1, 2, \ldots, 2n\}$ then there are always two which differ by* 1.

Solution:
Consider the groups $\{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\}$ as pigeon holes. The $n+1$ distinct integers form the pigeons and by pigeon hole principle, there exist a group g_k containing more than one integer; say $i, i+1 \in g_k$. This implies that among the $n+1$ distinct integers, there exist two $\{i, i+1\}$ which differ by 1.