Assignment 6 - Solutions

For the following questions, say whether the count is finite/countably infinite/uncountable with a rich justification. Do NOT oversimplify the problem by making trivial assumptions. Be unique in your answer. Wherever, the set is infinite, argue that the set is not finite.

1. - How many leaves are there in a Neem tree.
   - How many T-shirts are there on earth.
   - How many mosquitoes are there on earth.

   Ans: (i) Let us assume that the neem tree has no leaf completely covered by other leaves. That is, every leaf serves a purpose to capture sunlight at least $1mm^2$. Let xmm^2 area is occupied by the tree. Thus there are at most x leaves in the tree, and is finite in number.
   (ii) There are finite number of T-shirt manufacturing industries on the earth, and an industry could manufacture finite number of T-shirts. Thus there are finite number of T-shirts in the whole world.
   (iii) Mosquitoes breed from water. Finite quantity of water is available on earth. Therefore, the number of mosquitoes are finite.

2. Given a box of dimension $\sqrt{3} \times \sqrt{3} \times \sqrt{3}$ meters; Is the size of this box is finite or infinite.
   Ans: The size of the box is upper bounded by $2 \times 2 \times 2m^3$. Thus the size of the box is finite.

3. Consider a sorting program that takes an integer array of size n as an input;

   - If n is fixed, how many different inputs are possible (the number of different test cases).
   - If n is a variable, how many different inputs are possible (the number of different test cases).

   Ans: (i) if n is fixed say k, then the different inputs comes from the set is $N \times N \times \cdots \times N$ (k times). Since size of $N \times N \times \cdots \times N$ for any fixed number of times is countably infinite, there are countably infinite number of different test cases.
   (ii) If n is not fixed, then the different inputs are from sets N, $N \times N$, $N \times N \times N$, and so on. Thus the different inputs possible are from $P(N)$. Since the power set $P(N)$ is uncountable, the number of different test cases are uncountable.

4. Consider a three degree polynomial $a_3\ x^3 + a_2\ x^2 + a_1\ x^1 + a_0\ x^0$ with integer co-efficients; how many different three degree polynomials are possible?
   Ans: The number of three degree polynomials possible is equal to the cardinality of the set $I \times I \times I \times I$. Since the cardinality of the set $I \times I \times I \times I$ is countably infinite, there are countably infinite number of different three degree polynomials.

5. How many different sorting algorithms are possible?
   Ans: Note that any sorting algorithm takes a permutation of the n numbers as input. Each step of the sorting algorithm proceeds by changing the permutation from one to another, and finally obtaining the desired permutation. This can be modelled using a state diagram in which each state corresponds to a permutation, and the sorting algorithm starts from an input state and proceeds through a series of states before obtaining the final state. Since there are $n!$ possible states (permutations), any sorting algorithm corresponds to a path in the state diagram on $n!$ nodes. Further, we can have cycles in the above graph, which means some of the subpaths are computed repeatedly. This gives us a list of algorithms; $A_1, A_2, A_3, \ldots, A_k$ corresponds to paths in the graph, $A_{k+1}, A_{k+2}, \ldots$, are based on cycles in the graph. Thus, the number of sorting algorithms is countably infinite.

6. $A = \{$ set of C-programs $\}$; $B = \{$ set of C++ programs $\}$. Which set is bigger.
   Ans: Note that a C-program is an implementation of an algorithm. The same algorithm could be implemented in any other programming language say C++. Thus the number of programs in C is same as that of C++. Moreover the binary file corresponding to each C program is a string in $(0/1)^*$. Since $(0/1)^*$ is countably infinite, the number of C and C++ programs are countably infinite.

7. How many different word documents are possible?
Ans: Let $\sum$ be the set of keys available in the keyboard (set of all ASCII characters). It is easy to see that every word document is an element in $\sum^*$. Since $\sum^*$ is countably infinite, the number of word documents are countably infinite.

8. Compare the following sets

   - $A = [2, 6]$, $B = [0, 1]$
   - $A = (0, 1)$, $B = [0, 1]$
   - $A = [0, 1]$, $B = \mathbf{R}$

   Ans: For all three, we exhibit an injection from A to B and vice versa. By establishing 1-1 from A to B, we can conclude that $|A| \leq |B|$, and by the converse, we can conclude that $|B| \leq |A|$. Thus, we get $|A| = |B|$.

   (i) $A \rightarrow B$:
   $f(x) = \frac{1}{x}$.
   $B \rightarrow A$:
   $f(x) = 2 + (6 - 2)x$.
   Thus, we conclude that $|A| = |B|$.
   Note: Note $f(x) = 2 + (6 - 2)x$ is actually a bijection from $[0, 1]$ to $[2, 6]$ which can be generalized to any $[a, b]$. That is, $[0, 1]$ to $[a, b]$ is given by $f(x) = a + (b - a)x$.

   (ii) $f(x) = x$. This implies that $|A| \leq |B|$. Next, for each element $x \in B$ there exists $f(x) \in A$ where $f(x)$ is defined as follows.

$$f(x) = \begin{cases} x + \delta & \text{if } x \leq 0.5 \\ x - \delta & \text{if } x > 0.5 \end{cases} \quad \text{where } 0 < \delta < 0.5$$

   It follows that that $|B| \leq |A|$. Therefore, $|A| = |B|$.
   Aliter: B to A; $f(x) = \frac{1}{2} + (\frac{3}{4} - \frac{1}{2})x$.

   (iii) $A \rightarrow B$; $f(x) = x$. This implies that $|A| \leq |B|$.
   $B \rightarrow A$; Here, we divide R (i.e., $[-\infty, \infty]$ into four parts and map each to appropriate subinterval in $[0, 1]$ leaving some subinterval in $[0, 1]$.
   $f(x) = \frac{1}{2+x}$ if $x \in [1, \infty]$

   $f(x) = 0.4 + (0.1)x$ if $x \in [0, 1)$

   $f(x) = 0.6 - (0.1)x$ if $x \in (0, -1)$

   $f(x) = 0.8 + \frac{-1}{-10+x}$ if $x \in [-1, -\infty]$

   Note that the subinterval $(0.5, 0.6)$ does not have a pre-image.
   This implies that $|B| \leq |A|$. Thus we conclude that $|A| = |B|$.