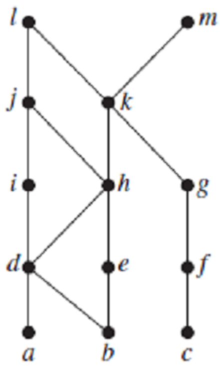
Assignment 4 : Relations - Solutions

1. For the Hasse diagram given below; find maximal, minimal, greatest, least, LB, glb, UB, lub for the subsets;
 (i) $\{d, k, f\}$



(ii) $\{b, h, f\}$
(iii) $\{d\}$
(iv) $\{a, b, c\}$
(v) $\{l, m\}$

Set	Greatest	Least	Maximum	Minimum	UB	LB	LUB	GLB
$\{d, k, f\}$	$\{k\}$	NIL	$\{k\}$	$\{d, f\}$	$\{k, l, m\}$	NIL	$\{k\}$	NIL
$\{b, h, f\}$	NIL	NIL	$\{h, f\}$	$\{b, f\}$	$\{l, m\}$	NIL	$\{k\}$	NIL
$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d, h, i, j, k, l, m\}$	$\{d, a, b\}$	$\{d\}$	$\{d\}$
$\{a, b, c\}$	NIL	NIL	$\{a, b, c\}$	$\{a, b, c\}$	$\{k, l, m\}$	NIL	$\{k\}$	NIL
$\{l, m\}$	NIL	NIL	$\{l, m\}$	$\{l, m\}$	NIL	$\{a, b, c, d, e$ $, f, g, h, k\}$	NIL	$\{k\}$

2. Does there exist a binary relation R such that $t(R)$ is infinite? Note: $t(R)$ is a transitive closure of R. Justify. Argue that if the underlying set is finite, then the algorithm that computes $t(R)$ terminates after finite steps.
Ans: Transitive closure of an infinite set is infinite. For proof, see Theorem 7 of scribe. Let R be a relation with finite number of elements, say $|R| = n$. Note that the transitive closure of R is $t(R) = \bigcup_{i=1}^{n} R^i$. Since R^i is finite for a fixed i, and could be computed in finite number of steps, it follows that $t(R)$ is also finite, and could be computed in finite number of steps.

3. Let R_1 and R_2 be relations on A. Prove or disprove; (i) $t(R_1 \cup R_2) \supset t(R_1) \cup t(R_2)$ (ii) Converse of (i).

Ans: Let $(a,b) \in t(R_1)$. Note that either $(a,b) \in R_1$ or there exist an element c in the set such that $(a,c),(c,b) \in R_1$. Observe that in both the cases, $(a,b) \in t(R_1 \cup R_2)$. Similar argument holds good with respect to the relation R_2. Thus we conclude that $t(R_1 \cup R_2) \supset t(R_1) \cup t(R_2)$. Converse is not true, which could be observed from the following example. Let $A = \{1,2\}$, $R_1 = \{(1,2)\}$, $R_2 = \{(2,1)\}$. Note that $t(R_1) = R_1$, $t(R_2) = R_2$, $t(R_1 \cup R_2) = \{(1,2),(2,1),(1,1),(2,2)\}$. Thus $t(R_1 \cup R_2) \not\subset t(R_1) \cup t(R_2)$.

4. Note that Bell's number (the number of equivalence relations) acts as a powerful lower bound for the number of transitive relations. Present either a lower bound (different from B_n) or an upper bound (different from 2^{n^2}) with a suitable justification. Present a good bound with a rich argument.
Ans: All partial ordered relations are transitive realtions. Moreover, all total ordered relations are partial order. The number of total ordered relations on a set of size n is at most $n!$. Thus the number of transitive relations is at least $n!$.

5. Argue that $B_n \leq 2^{n^2}$. Choose a suitable proof technique.
Ans: Note that $B_n = \sum\limits_{k=0}^{n-1} \binom{n-1}{k} B_{n-1-k}$.
Proof by induction:
Base case: when $n = 0$, $B_0 = 1 \leq 2^0$.
Induction Hypothesis: for every $k \geq 0$, let $B_k \leq 2^{k^2}$.
Induction step: Consider B_{k+1}, $k \geq 0$. From the definition,
$B_{k+1} = \sum\limits_{i=0}^{k} \binom{k}{i} B_{k-i} = \binom{k}{0} B_k + \binom{k}{1} B_{k-1} + \cdots + \binom{k}{k} B_0$
$\leq B_k(\binom{k}{0} + \binom{k}{1} + \cdots + \binom{k}{k})$
$\leq B_k 2^k$. From induction hypothesis, $B_k \leq 2^{k^2}$. Thus, $B_{k+1} \leq 2^{k^2} 2^k$
$B_{k+1} \leq 2^{(k+1)^2}$. □

6. Present a Hasse diagram (or a poset) and an associated subset for each of the following; you may choose to present a different Hasse diagram if you wish so

   - a subset such that it has two maximal and two minimal elements.
   - a subset such that it has a maximal element but no minimal elements. Is it possible to find a such a subset if the underlying set is an infinite set.
   - a subset such that it has a lower bound but no greatest lower bound. If such a subset is not possible, argue why?
   - a subset such that it has an upper bound but no least upper bound. If such a subset is not possible, argue why?

   Ans: Conisder the following hasse diagram.

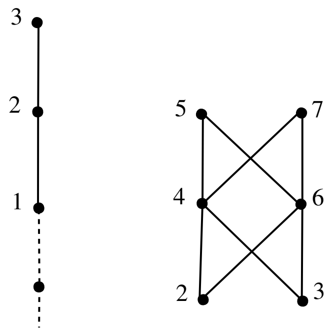


Fig a Fig b

(i) In Fig b, for the subset $\{4, 6\}$, maximal elements are $\{4, 5\}$ and minimal elements are $\{4, 5\}$.
(ii) In Fig a, for the subset $\{3, 2, 1, \ldots\}$, 3 is the maximal element.
(iii) In Fig b, consider the subset $\{4, 6\}$. Lower bounds are $\{2, 3\}$, however there is no greatest lower bound.
(iv) In Fig b, consider the subset $\{4, 6\}$. Upper bounds are $\{5, 7\}$, however there is no least upper bound.

7. Is there finite set such that it is a poset and totally ordered set but not a well-ordered set. Justify.
   Ans: $(A, \preceq)$ is a *well order* if $(A, \preceq)$ is a total order and for all $A' \subseteq A, A' \neq \phi, A'$ has a least element. Thus all finite totally ordered sets are well ordered.

8. How many different partial orders are possible on a set $\{1, \ldots, n\}$. If not a precise number, present a suitable lower bound or upper bound with a justification.
   Ans: All totally ordered relations are partial order. Since there are at most $n!$ totally ordered relations, there exists at least $n!$ partial orders. Note that there exists a hasse diagram corresponding to each partial order. Moreover, if two partial orders are same then they have the same hasse diagram. Thus the number of partial orders is precisely the number of different hasse diagrams possible. Since hasse diagrams are all forests (collection of trees), the total number of partial orders is the number of different forests possible. Also we could come up with a lower bound on the number of partial orders as the maximum number of labelled trees, which is n^{n-2}.