Discrete Mathematics - Assignment III Solutions

Proof by Mathematical Induction. (for questions 1-9)

1. Show that $nC_r = \frac{n!}{(n-r)!r!}$.

   Solution: Let us prove this by induction on n. As mentioned before, we focus on n and do not care about r. Further, the proof works for any r.
   Base Case: If $n = 2$ then, $r = 1$. The number of 1-size subsets on an 2−element set is 2. $P(2)$ is true.
   Hypothesis: Assume that, the statement is true for n, $n \geq 2$.
   Induction Step: For $n + 1$, $n \geq 2$.
   We know that $(n+1)C_r = n_{c_{r-1}} + n_{c_r}$
   By the induction hypothesis,
   $= \frac{n!}{(n-(r-1))!(r-1)!} + \frac{n!}{(n-r)!r!}$
   $= \frac{n!}{(n-(r-1))(n-r)!(r-1)!} + \frac{n!}{(n-r)!r(r-1)!}$
   $= \frac{n!}{(n-r)!(r-1)!}\left(\frac{1}{(n-(r-1))} + \frac{1}{r}\right)$
   $= \frac{n!}{(n-r)!(r-1)!}\left(\frac{n+1}{r(n-r+1)}\right)$
   $= \frac{(n+1)n!}{((n+1)-r)!r(r-1)!}$
   $= \frac{(n+1)!}{((n+1)-r)!r!}$
   Hence, $\forall n \geq 2$, $P(n+1)$ is true if $P(n)$ is true.

2. Show that there are 3^n ternary strings.
   Solution: Let us prove this statement by the mathematical induction on the length of the string, n. A ternary string is a sequence of digits, where each digit is either 0, 1, or 2.
   Base Case: $n = 1$. There are three possibilities. Therefore, $3^1 = 3$ ternary strings exist of length one.
   Hypothesis: Assume that the given statement is true for all $n = k, k \geq 1$. i.e., there are 3^k ternary strings of length k.
   Induction Step: Let $n = k + 1, k \geq 1$.
   Let the string be $a_1a_2 \ldots a_ka_{k+1}$, where each $a_i \in \{0, 1, 2\}$.
   By the hypothesis, $a_1a_2 \ldots a_k$ has got 3^k possibilities and the position a_{k+1} has got three possibilities. In total, there are $3^k \cdot 3^1$ possibilities. i.e., there are 3^{k+1} ternary strings of length $k + 1$.

3. Show that in any group of n ($n \geq 4$) people there exist a pair of friends or there exist three mutual enemies.

   Solution:

   Base Case: $n = 4$. All possibilities for a group four people are as follows.

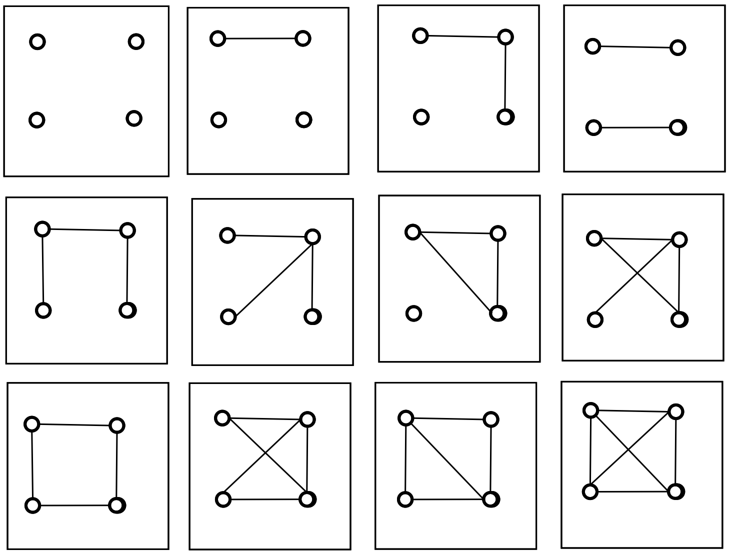

Figure 1: Nodes denotes the people and the edge between two nodes says they are friends.

Hypothesis: Assume that the statement is true for $n = k, k \geq 4$.
Induction Step: Let $n = k + 1, k \geq 4$. By hypothesis, in any group of k ($k \geq 4$) people there exist a pair of friends or there exist three mutual enemies. Adding a person in this group will not affect the existing three mutual enemies or a pair of friends. Hence proved.

4. Show that for every n, there are more than n prime numbers.
Solution:
Base case: $n = 1$. $\{2, 3, \ldots\}$ are prime integers. Clearly, for the integer '1', there exist more than one.
Induction hypothesis: Assume for $n = k, k \geq 1$, that there exist more than k prime integers. Let the prime numbers be $p_1, p_2, \ldots, p_k, p_{k+1} \ldots$.
Induction step: We claim that for $n = k + 1$, $k \geq 1$ there exist more than $k + 1$ prime numbers. Consider the number $P = p_1 \cdot p_2 \ldots p_k \cdot p_{k+1} + 1$, i.e. P is one plus the product of the prime numbers $p_1, p_2, \ldots, p_{k+1}$.
We consider the following cases to complete the proof.
Case a: If P is a prime number, then there exist more than $k + 1$ prime numbers with $(k + 2)^{nd}$ prime number being P.
i.e., $\{p_1, p_2, \ldots, p_k, p_{k+1}, P\}$ are the set of $(k + 2)$ prime numbers.
Case b: If P is not a prime number, then note that there exist a prime factorization for P and none of $\{p_1, p_2, \ldots, p_k, p_{k+1}\}$ are its prime factors. This implies that there exist a prime factor p_{k+2} for P such that $p_{k+2} \neq p_i$, $1 \leq i \leq k + 1$. Therefore, $\{p_1, p_2, \ldots, p_k, p_{k+1}, p_{k+2}\}$ are prime numbers with cardinality more than $k + 1$. The induction is complete and hence the claim follows.

5. Show that $\left(\frac{n}{e}\right)^n \leq n!$
Solution:
Base Case: The statement is true for $n = 0$.

Hypothesis: Assume that the statement is true for $n = k$, $k \geq 0$. i.e., $\left(\frac{k}{e}\right)^k \leq k!$.
Induction Step: Let $n = k+1, k \geq 0$

$$\begin{aligned}
(k+1)! &\geq (k+1) \cdot \left(\frac{k}{e}\right)^k \text{ (by hypothesis)} \\
&= (k+1) \cdot \left(\frac{k}{k+1}\right)^k \cdot \left(\frac{k+1}{e}\right)^k \\
&= \left(\frac{(k+1)^{k+1}}{e^k}\right) \cdot \frac{k^k}{(k+1)^k} \\
&\geq \left(\frac{(k+1)^{k+1}}{e^k}\right) \cdot \frac{1}{e} \text{ (Since for any } k, \ \left(1+\frac{1}{k}\right)^k \leq e) \\
&= \left(\frac{k+1}{e}\right)^{k+1}
\end{aligned}$$

6. Show that for each integer $n \geq 1$, the n^{th} Fibonacci number F_n is less than $\left(\frac{13}{8}\right)^n$.
Solution: The n^{th} Fibonacci number is $F_n = F_{n-1} + F_{n-2}$.
Base Case: $n = 2$. $F_2 = F_1 + F_0 = 1 + 1 = 2 < \left(\frac{13}{8}\right)^2 = 2.640625$
Hypothesis: Assume that the statement is true for $n = k$, $k \geq 2$. i.e., $F_k < \left(\frac{13}{8}\right)^k$. This implies, $F_{k-1} + F_{k-2} < \left(\frac{13}{8}\right)^k$.

$$\begin{aligned}
F_n = F_{k+1} &= F_k + F_{k-1} \\
&= F_{k-1} + F_{k-2} + F_{k-1} \\
&< \left(\frac{13}{8}\right)^k + F_{k-1} \text{ (by hypothesis)} \\
&< \left(\frac{13}{8}\right)^k + \left(\frac{13}{8}\right)^{k-1} \text{ (by hypothesis)} \\
&= \left(\frac{13}{8}\right)^{k-1} \left[\frac{13}{8} + 1\right] \\
&< \left(\frac{13}{8}\right)^{k-1} \left(\frac{13}{8}\right)^2 \\
&= \left(\frac{13}{8}\right)^{k+1}
\end{aligned}$$

Thus, F_n is less than $\left(\frac{13}{8}\right)^n$, for all $n \geq 2$.

7. For each integer $n \geq 2$, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}$
Solution:
Base Case: The statement is true for $n = 2$ (Since, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} > \sqrt{2}$).
Hypothesis: Assume that the statement is true for $n = k$, $k \geq 2$. i.e., $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} > \sqrt{k}$

Induction Step: Let $n = k+1, k \geq 2$.

$$\begin{aligned}
\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} &> \sqrt{k} + \frac{1}{\sqrt{k+1}} \text{ (by hypothesis)} \\
&= \frac{\sqrt{k}\sqrt{k+1}+1}{\sqrt{k+1}} \\
&> \frac{\sqrt{k}\sqrt{k}+1}{\sqrt{k+1}} \\
&= \frac{k+1}{\sqrt{k+1}} \\
&= \sqrt{k+1}
\end{aligned}$$

8. $(x+y)$ is a factor of the polynomial $x^{2n+1} + y^{2n+1}$.
Solution:
Base Case: $n = 0$. $(x+y)$ is a factor of the polynomial $x^1 + y^1$
Hypothesis: Assume that the statement is true for $n = k$, $k \geq 1$. i.e., $(x+y)$ is a factor of the polynomial $x^{2k+1} + y^{2k+1}$.
Induction Step: Let $n = k+1, k \geq 1$

$$\begin{aligned}
x^{2(k+1)+1} + y^{2(k+1)+1} &= x^{(2k+1)+2} + y^{(2k+1)+2} \\
&= x^{(2k+1)} \cdot x^2 + y^{(2k+1)} \cdot y^2 \\
&= x^{(2k+1)} \cdot x^2 + y^{(2k+1)} \cdot y^2 + y^{(2k+1)} \cdot x^2 - y^{(2k+1)} \cdot x^2 \\
&= y^{(2k+1)}(y^2 - x^2) + x^2(y^{(2k+1)} + x^{(2k+1)})
\end{aligned}$$

$(x+y)$ is a factor of $(y^{2k+1} + x^{2k+1})$ (by hypothesis) and also the factor of $(y^2 - x^2)$. It follows that, $(x+y)$ is the factor of the polynomial $x^{2(k+1)+1} + y^{2(k+1)+1}$.

Hence, $(x+y)$ is a factor of the polynomial $x^{2n+1} + y^{2n+1}$.

9. $11^{n+2} + 12^{2n+1}$ is divisible by 133.
Solution:
Base Case: $n = 0$. $11^2 + 12^1 = 133$, which is divisible by 133.
Hypothesis: Assume that the statement is true for $n = k$, $k \geq 1$. i.e., $11^{k+2} + 12^{2k+1}$ is divisible by 133.
Induction Step: Let $n = k+1, k \geq 1$.

$$\begin{aligned}
11^{(k+1)+2} + 12^{2(k+1)+1} &= 11^{(k+2)+1} + 12^{2k+1+2} \\
&= 11^{(k+2)+1} + 12^{2k+1} \cdot 12^2 \\
&= 11^{(k+2)+1} + 12^{2k+1} \cdot (133 + 11) \\
&= 11 \cdot (11^{(k+2)} + 12^{2k+1}) + 133 \cdot 12^{2k+1}
\end{aligned}$$

$(11^{(k+2)} + 12^{2k+1})$ is divisible by 133 by the hypothesis. It follows that, $11 \cdot (11^{(k+2)} + 12^{2k+1}) + 133 \cdot 12^{2k+1}$ is divisible by 133. Thus, $11^{(k+1)+2} + 12^{2(k+1)+1}$ is divisible by 133.

Hence, $11^{n+2} + 12^{2n+1}$ is divisible by 133 for all $n \geq 0$.

10. A monkey is asked to climb up a ladder having n-steps. Each climb is such that the monkey takes either one step or two steps. i.e., from Step-1, it can go to Step-2 or Step-3. From a Step-i, it can go to Step-$(i+1)$ or Step-$(i+2)$. In how many different ways can a monkey climb up the ladder.
Solution:
The person can reach n^{th} stair from either $(n-1)^{th}$ stair or from $(n-2)^{th}$ stair. Let the total number of ways to reach n^{th} stair be '$ways(n)$'. Thus, $ways(n) = ways(n-1) + ways(n-2), n \geq 3$ with the boundary cases $ways(1) = 1$ (There is only one way to climb stair 1) and $ways(2) = 2$ (There are two ways to climb stair 2).

    (i) Present a precise bound and prove your answer using Mathematical Induction. If you think obtaining a precise bound is challenging, then present a meaningful lower bound and an upper bound, and prove both of them.

    Proof technique: your choice.
    Solution:
    (i) The given sequence is precisely the Fibonacci sequence. So, the upper bound is as in *Problem 6*
    (ii) Tight Bound: $(1+\sqrt{5})/2$, Golden Ratio
    (iii) Lower Bound: n

11. Show that $\sqrt{3}$ is irrational. (Hint: Proof by contradiction)
Solution:
On the contrary, assume that $\sqrt{3}$ is rational, then $\sqrt{3} = \frac{a}{b}$ for some $a, b \in \mathbb{N}$ and $b \neq 0$. Note that, $\frac{a}{b}$ is the simplest form.

$$\sqrt{3} = \tfrac{a}{b}$$
$$\Rightarrow 3 = \tfrac{a^2}{b^2}$$
$$\Rightarrow a^2 = 3 \cdot b^2$$

a^2 is a multiple of 3 and hence, a is also a multiple of 3 (If a is not a multiple of 3, then a is of the form either $3k+1$ or $3k+2$. Thus, a^2 is of the form either $9k^2+6k+1$ or $9k^2+12k+4$, which is a contradiction to our assumption that a^2 is a multiple of 3). So, assume that $a = 3k$, for some $k \in \mathbb{N}$. Thus,

$$a^2 = 3 \cdot b^2 \Rightarrow 9k^2 = 3b^2 \Rightarrow 3k^2 = b^2 \Rightarrow b = \pm 3k$$

Hence, b is also a multiple of 3. In this case $\frac{a}{b}$ is not in simplest form, which is a contradiction.

12. Suppose that the 10 integers $1, 2, \ldots, 10$ are randomly positioned around a circular wheel. Show that there are consecutive three numbers whose sum is at least 17. (Hint: Proof by contradiction)
Solution: Proof by contradiction:
There are 10 triples of adjacent numbers with sums $S_1, S_2, \ldots, S_{10}$. If each is less than 17, they all add up to at most $16 \times 10 = 160$. However,

in the latter sum each of the numbers $1, 2, \ldots, 10$ appears 3 times, so that the sum must be at least $3 \times 55 = 165$ ($55 = 1 + 2 + ... + 10$), which is a contradiction. It follows that our assumption is false.