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COM 205T Discrete Structures for Computing

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Solutions - Assignment II

1. Write the definition of 'Prime number' in first order logic.
   Solution:
   Definition of Prime Number: A prime number is a natural number greater than 1 that has no positive divisors other than 1 and itself.
   FOL: 'x is prime' is definable in $\mathbb{N}$ by $(1 < x) \wedge \forall y((y|x) \rightarrow ((y = 1) \vee (y = x)))$, where $y|x$ mean $\exists z(y \cdot z = x)$

   (or)

   $\forall x[((1 < x) \wedge \forall y((y|x) \rightarrow ((y = 1) \vee (y = x)))) \rightarrow Prime(x)]$

2. Negate the following: $\forall x \exists \epsilon((x > 0 \wedge \epsilon > 0) \wedge \forall y(y > 0 \rightarrow x - y \geq \epsilon))$.
   Solution:
   $\exists x \forall \epsilon((x \leq 0 \vee \epsilon \leq 0) \vee \exists y(y > 0 \wedge (x - y) < \epsilon))$

   (or)

   $\exists x \forall \epsilon((x > 0 \wedge \epsilon > 0) \rightarrow \exists y(y > 0 \wedge (x - y) < \epsilon))$

3. Prove or Disprove:

   (a) $\exists x(P(x) \wedge Q(x)) \rightarrow \exists x P(x) \wedge \exists x Q(x)$
   Solution:

	$\exists x(P(x) \wedge Q(x))$	...	(1)
Proof			
From 1 :	$P(a) \wedge Q(a)$	...	(2) – Existential Instantiation
2 :	$P(a)$	...	(3)
2 :	$Q(a)$	...	(4)
3 :	$\exists x P(x)$	...	(5) – Existential Generalization of (3)
4 :	$\exists x Q(x)$	...	(6) – Existential Generalization of (4)
3, 4 :	$\exists x P(x) \wedge \exists x Q(x)$		*QED*

   (b) $\exists x P(x) \wedge \exists x Q(x) \rightarrow \exists x(P(x) \wedge Q(x))$

   Solution:
   The above implication is false. *Counter Example:* UOD: $\mathbb{N}$. $P(x): x = 2$ and $Q(x): x = 3$. The premise is true and the conclusion is false. Therefore the above statement is false.

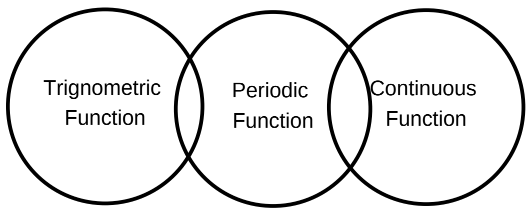
4. Prove or Disprove:

   (a) $[\exists x P(x) \rightarrow \forall x Q(x)] \rightarrow \forall x[P(x) \rightarrow Q(x)]$
   Solution:
   $[\exists x P(x) \rightarrow \forall x Q(x)]$
   $\leftrightarrow [\neg \exists x P(x) \vee \forall x Q(x)]$

$\leftrightarrow [\forall x \neg P(x) \vee \forall x Q(x)]$
$\rightarrow [\forall x(\neg P(x) \vee Q(x))]$
$\rightarrow [\forall x(P(x) \rightarrow Q(x))]$. QED.

(b) $\forall x[P(x) \rightarrow Q(x)] \rightarrow [\exists x P(x) \rightarrow \forall x Q(x)]$
Solution:
The given implication is false. *Counter Example:* UOD: Set of integers. Let $P(x)$ be the statement "x is divisible by 4". Let $Q(x)$ be the statement "x is divisible by 2". Thus, the premise is true and the conclusion is false. Therefore the above statement is false.

5. Check the validity of the argument.
Some trigonometric functions are periodic. Some periodic functions are continuous. Therefore, some trigonometric functions are continuous.
Solution:
The given conclusion is false. The following Venn diagram is a counter example for the given conclusion.



6. Check the validity of the argument.
All clear explanations are satisfactory. Some excuses are unsatisfactory. Hence some excuses are not clear explanations.
Solution:
The conclusion is true by the following argument.

Premise:	$\forall x(C(x) \rightarrow S(x))$	...	(1)
Premise:	$\exists x(E(x) \wedge \neg S(x))$	...	(2)
1 :	$C(a) \rightarrow S(a)$	...	(3) – Universal Instantiation
3 :	$\neg S(a) \rightarrow \neg C(a)$	...	(4) – Contrapositive of (3)
2 :	$E(a) \wedge \neg S(a)$	...	(5) – Existential Instantiation
5 :	$E(a)$	...	(6)
5 :	$\neg S(a)$	...	(7)
4, 7 :	$\neg C(a)$	...	(8)
6, 8 :	$E(a) \wedge \neg C(a)$	...	(9)
9, *Conclusion:*	$\exists x(E(x) \wedge \neg C(x))$		Existential Generalization.

7. Let the universe of discourse be the set of integers. For each of the following assertions, find a predicate P which makes the implication false.

- $\forall x \exists! y P(x,y) \rightarrow \exists! y \forall x P(x,y)$
Solution:
Let $P(x,y)$ be the statement $x + y = 0$. Thus, the truth value of $\forall x \exists! y P(x,y)$ is true (Since, for every integer x there exist an integer $-x$ such that $x + (-x) = 0$) and the truth value of

$\exists! y \forall x P(x,y)$ is false (Since, there does not exist an integer y such that $\forall x \in \mathbb{N}$, $x + y = 0$). Therefore, the implication is false for the given predicate.

- $\exists! y \forall x P(x,y) \rightarrow \forall x \exists! y P(x,y)$
  Solution:
  Let $P(x,y)$ be the statement $x \cdot y = 0$. Thus, the truth value of $\exists! y \forall x P(x,y)$ is true (Since, there exist an integer $y = 0$ such that $\forall x \in \mathbb{N}$, $x \cdot y = 0$) and the truth value of $\forall x \exists! y P(x,y)$ is false (Since, when $x = 0$, $x \cdot y = 0$ for all values of y). Therefore, the implication is false for the given predicate.

8. Prove or Disprove: $\forall x(P(x) \vee Q(x)) \rightarrow \forall x P(x) \vee \exists x Q(x)$
   Solution:
   Proof by contradiction: Assume on the contrary that the conclusion is FALSE. i.e., include $\neg$ *Conclusion* as part of premise.

premise	$\forall x\ (P(x) \vee Q(x))$	...	(1)
premise assumed	$\neg[\forall x\ P(x) \vee \exists x\ Q(x)]$	...	(2)
2	$\neg\forall x\ P(x) \wedge \neg\exists x\ Q(x)$	...	(3)
3	$\exists x\ \neg P(x) \wedge \forall x\ \neg Q(x)$	...	(4)
4	$\exists x\ \neg P(x)$	...	(5)
EI of 5	$\neg P(a)$	...	(6)
4	$\forall x\ \neg Q(x)$	...	(7)
UI of 7	$\neg Q(a)$	...	(8)
7, 8	$\neg P(a) \wedge \neg Q(a)$	...	(9)
9	$\neg[P(a) \vee Q(a)]$	...	(10)
UI of 1	$P(a) \vee Q(a)$	...	(11)
10, 11	$\neg[P(a) \vee Q(a)] \wedge [P(a) \vee Q(a)]$		*a contradiction*

Therefore our assumption is wrong/FALSE and conclusion is TRUE. Therefore $\forall x\ P(x) \vee \exists x\ Q(x)$ follows from $\forall x\ (P(x) \vee Q(x))$.

(or)

Premise $\forall x(P(x) \vee Q(x)) \rightarrow \forall x P(x) \vee \exists x Q(x)$
$\leftrightarrow \neg\forall x(P(x) \vee Q(x)) \vee (\forall x P(x) \vee \exists x Q(x))$
$\leftrightarrow \exists x \neg(P(x) \vee Q(x)) \vee (\forall x P(x) \vee \exists x Q(x))$
$\leftrightarrow \exists x(\neg P(x) \wedge \neg Q(x)) \vee (\forall x P(x) \vee \exists x Q(x))$
$\leftrightarrow ((\exists x \neg P(x)) \wedge (\exists x \neg Q(x))) \vee (\forall x P(x) \vee \exists x Q(x))$
$\leftrightarrow ((\neg\forall x P(x)) \wedge (\neg\forall x Q(x))) \vee (\forall x P(x) \vee \exists x Q(x))$

$\forall x P(x)$, $\forall x Q(x)$ and $\exists x Q(x)$ are atomic predicates. Therefore, we can check the validity of the above proposition using truth table.

A	B	C			D	E	
$\forall x P(x)$	$\forall x Q(x)$	$\exists x Q(x)$	$\neg A$	$\neg B$	$\neg A \wedge \neg B$	$A \vee C$	$D \vee E$
1	1	1	0	0	0	1	1
1	1	0 (NA)	(NA)	(NA)	(NA)	(NA)	(NA)
1	0	1	0	1	0	1	1
1	0	0	0	1	0	1	1
0	1	1	1	0	0	1	1
0	1	0 (NA)	(NA)	(NA)	(NA)	(NA)	(NA)
0	0	1	1	1	1	1	1
0	0	0	1	1	1	0	1

Since, the last column forms a tautology, the given proposition is true.